HOMOTOPY LIE ALGEBRA OF CLASSIFYING SPACES FOR HYPERBOLIC COFORMAL 2-CONES

J.-B. GATSI\-N\-ZI

(communicated by Johannes Huebschmann)

Abstract

In this paper, we show that the rational homotopy Lie algebra of classifying spaces for certain types of hyperbolic coformal 2-cones is not nilpotent.

1. Introduction

A simply connected space $X$ is called an $n$-cone if it is built up by a sequence of cofibrations

$$
Y_k \xrightarrow{f_k} X_{k-1} \xrightarrow{j_k} X_k
$$

with $X_0 = *$ and $X_n \simeq X$. One can further assume that $Y_k \simeq \Sigma^{k-1}W_k$ is a $(k - 1)$-fold suspension of a connected space $W_k$ [3]. In particular a 2-cone $X$ is the cofibre of a map between two suspensions

$$
\Sigma A \xrightarrow{f} \Sigma B \to X.
$$

Spaces under consideration are assumed to be 1-connected and of finite type, that is, $H^i(X; \mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space. To every space $X$ corresponds a free chain Lie algebra of the form $(L(V), \delta)$ [2], called a Quillen model of $X$. It is an algebraic model of the rational homotopy type of $X$. In particular, one has an isomorphism of Lie algebras $H_*(L(V), \delta) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. The model is called minimal if $\delta V \subset L^{>2}(V)$. A space $X$ is called coformal if there is a map of differential Lie algebras $(L(V), \delta) \to (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$ that induces an isomorphism in homology.

Any continuous map $f : X \to Y$ has a Lie representative $\tilde{f} : (L(W), \delta') \to (L(V), \delta)$ between respective models of $X$ and $Y$.

If $X$ is a 2-cone as defined by (1) and $\tilde{f} : L(W) \to L(V)$ is a model of $f$, then a Quillen model of the cofibre $X$ of $f$ is obtained as the push out of the following diagram:

$$
\begin{array}{ccc}
(L(W), 0) & \xrightarrow{\tilde{f}} & (L(V), 0) \\
\downarrow \uparrow & & \downarrow \uparrow \\
(L(W \oplus sW), d) & \overset{\overset{\tilde{f}}{\longrightarrow}}{\xrightarrow{\quad}} & (L(V \oplus sW), \delta)
\end{array}
$$

This work was supported by the Abdus Salam ICTP in cooperation with SIDA.

Received November 14, 2003, revised April 21, 2004; published on May 3, 2004.

2000 Mathematics Subject Classification: Primary 55P62; Secondary 55M30.

Key words and phrases: rational homotopy, coformal spaces, 2-cones, differential Ext.

where \((L(W \oplus sW), d)\) is acyclic. Moreover the differential on \(L(V \oplus sW)\) verifies \(\delta sW \subset L(V)\). Hence a 2-cone \(X\) has a Quillen model of the form \((L(V_1 \oplus V_2), \delta)\) such that \(\delta V_1 = 0\) and \(\delta V_2 \subset L(V_1)\).

A Sullivan model of a space \(X\) is a cochain algebra \((\wedge Z, d)\) that algebraically models the rational homotopy type of \(X\). In particular, one has an isomorphism of graded algebras \(H^*(\wedge Z, d) \cong H^*(X; \mathbb{Q})\). The model is called minimal if \(dZ \subset \wedge^2 Z\).

In this case the vector spaces \(Z^n\) and \(\text{Hom}(\pi_n(X), \mathbb{Q})\) are isomorphic. If \(X\) has the rational homotopy type of a finite CW-complex, we say that \(X\) is elliptic if \(Z\) is finite dimensional, otherwise \(X\) is called hyperbolic.

2. Models of classifying spaces

Henceforth \(X\) will denote a simply connected finite CW-complex and \(L_X\) its homotopy Lie algebra. Let \(\text{out} X\) denote the space of free self homotopy equivalences of \(X\), \(\text{aut}_1(X)\) the path component of \(\text{out} X\) containing the identity map of \(X\). The space \(\text{Baut}_1(X)\) classifies fibrations with fibre \(X\) over simply connected base spaces [4].

The Schlessinger-Stasheff model for \(\text{Baut}_1(X)\) is defined as follows [12]. If \((L(V), \delta)\) is a Quillen model of \(X\), we define a differential Lie algebra \(\text{Der} L(V) = \oplus_{k \geq 1} \text{Der}_k L(V)\) where \(\text{Der}_k L(V)\) is the vector space of derivations of \(L(V)\) which increase the degree by \(k\), with the restriction that \(\text{Der}_1 L(V)\) is the vector space of derivations of degree 1 that commute with the differential \(\delta\).

Define the differential Lie algebra \((sL(V) \oplus \text{Der} L(V), D)\) as follows:

- The graded vector space \(sL(V) \oplus \text{Der} L(V)\) is isomorphic to \(sL(V) \oplus \text{Der} L(V)\),
- If \(\theta, \gamma \in \text{Der} L(V)\) and \(sx, sy \in sL(V)\), then \([\theta, \gamma] = \theta \gamma - (-1)^{[\theta][\gamma]} \gamma \theta\), \([\theta, sx] = (-1)^{[\theta]} s\theta(x)\) and \([sx, sy] = 0\),
- The differential \(D\) is defined by \(D\theta = [\delta, \theta], D(sx) = -s\delta x + ad x\), where \(ad x\) is the inner derivation determined by \(x\).

From the Sullivan minimal model \((\wedge Z, d)\), Sullivan defines the graded differential Lie algebra \((\text{Der} \wedge Z, D)\) as follows [13]. For \(k > 1\), the vector space \((\text{Der} \wedge Z)_k\) consists of the derivations on \(\wedge Z\) that decrease the degree by \(k\) and \((\text{Der} \wedge Z)_1\) is the vector space of derivations of degree 1 verifying \(d\theta + \theta d = 0\). For \(\theta, \gamma \in \text{Der} \wedge V\), the Lie bracket is defined by \([\theta, \gamma] = \theta \gamma - (-1)^{[\theta][\gamma]} \gamma \theta\) and the differential \(D\) is defined by \(D\theta = [d, \theta]\).

We have the following result:

Theorem 1. [13, 12, 14] The differential Lie algebras \((\text{Der} \wedge Z, D)\) and \((sL(V) \oplus \text{Der} L(V), D)\) are models of the classifying space \(\text{Baut}_1(X)\).

An indirect proof of the Schlessinger-Stasheff model is given in [8, Theorem 2].

3. The classifying space spectral sequence

Recall that if \((L, \delta)\) is a graded differential Lie algebra, then \(L\) becomes an \(UL\) module by the adjoint representation \(ad : L \to \text{Hom}(L, L)\). In the sequel all Lie
algebras are endowed with the above module structure.

Let \((L(V), \delta)\) be a Quillen model of a finite CW-complex and \((TV, d)\) its enveloping algebra. On the TV-module \(TV \otimes (Q \oplus sV)\), define a \(Q\)-linear map

\[
S : TV \otimes (Q \oplus sV) \to TV \otimes (Q \oplus sV)
\]
as follows:

- \(S(1 \otimes x) = 0\) for all \(x \in Q \oplus sV\),
- \(S(v \otimes 1) = 1 \otimes sv\) for all \(v \in V\),
- If \(a \in TV\) and \(x \in TV \otimes (Q \oplus sV)\) with \(|x| > 0\), then \(S(a \cdot x) = (-1)^{|a|}a \cdot S(x)\).

The differential on the TV-module \(TV \otimes (Q \oplus sV)\) is defined by

\[
D(1 \otimes sv) = v \otimes 1 - S(dv \otimes 1) \text{ for } v \in V \text{ and } D(1 \otimes 1) = 0.
\]

It follows from [1] that \((TV \otimes (Q \oplus sV), D)\) is acyclic, hence it is a semifree resolution of \(Q\) as a \((TV, d)\)-module [6, §6].

Using the Schlessinger-Stasheff model of the classifying space, the author proved the following:

Theorem 2. [8] The differential graded vector spaces \(\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V))\) and \(sL(V) \otimes \text{Der} L(V)\) are isomorphic. Moreover, for \(n \geq 0\), the \(Q\)-vector spaces \(\text{Ext}^n_{TV}(Q, L(V))\) with \(\pi_{n+1}(U_{\text{aut}_1} X) \otimes Q\) are isomorphic.

In particular if \(X\) is a coformal space, one has an isomorphism \(\pi_n(B\text{aut}_1 X) \otimes Q \cong \text{Ext}^n_{U_{\text{aut}_1} X}(Q, L_X)\). Therefore \(\pi_n(B\text{aut}_1 X) \otimes Q\) can be computed by the means of a projective resolution of \(Q\) as an \(UL_X\)-module.

Consider the complex \(P = \text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V))\). Filter \(V\) as follows

\[
F_0V = 0, \quad F_{p+1}V = \{x \in V : dx \in L(F_pV)\}.
\]

We will denote \(V_p = F_pV/F_{p-1}V\) if \(F_{n-1}V \neq F_n V = V\), following Lemaire [10] we say that \(V\) is of length \(n\). We will restrict to spaces with a Quillen model of length \(n\).

Define a filtration on \(P = TV \otimes (Q \oplus sV)\) as follows:

\[
P_0 = TV \otimes Q, \quad P_1 = TV \otimes (Q \oplus sV_1), \ldots, P_n = TV \otimes (Q \oplus sV_{\leq n}).
\]

We filter the complex

\[
\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V))
\]

by

\[
F_k = \{f : f(F_{k-1}) = 0\}.
\]

This yields a spectral sequence \(E_r\) such that \(E_r^{p,q} = \text{Hom}_Q(sV_p, L_X)\) for \(p > 1\), \(E_1^{0,q} = \text{Hom}_Q(Q, L_X)\) and that converges to \(\text{Ext}^r_{TV}(Q, L(V))\). This sequence will be called the classifying space spectral sequence of \(X\).

Now assume that \(X\) is coformal and let \(A = Ul_X\). If \(L(V_1)/I\) is a minimal presentation of \(L_X\), then there is a quasi-isomorphism \((L(V_1 \oplus V_2 \oplus \cdots \oplus V_n), \delta) \to L_X\) which extends to \(p : (TV, d) \xrightarrow{\sim} (A, 0)\). The \((E_1, d)\) term provides a resolution

\[
\cdots \to A \otimes sV_n \to A \otimes sV_{n-1} \to \cdots \to A \otimes sV_1 \to A \to Q
\]
of $Q$ as an $A$-module. Here the differential is given by the composition

$$sV_n \xrightarrow{D} TV \otimes (Q \oplus sV_{n-1}) \xrightarrow{p \otimes id} A \otimes (Q \oplus sV_{n-1}).$$

The spectral sequence will therefore collapse at $E_2$ level. Moreover $\operatorname{Ext}_A^*(Q, C_X)$ is endowed with a Lie algebra structure verifying

$$[\operatorname{Ext}_A^p(\cdot, \cdot), \operatorname{Ext}_A^q(\cdot, \cdot)] \subset \operatorname{Ext}_A^{p+q-1}(\cdot, \cdot).$$

The Lie bracket can be defined using the bijection between the Koszul complex $C^*(C_X, \Lambda_X)$ and derivations on the Sullivan model $C^*(C_X, Q)$ of $X$ [9, Proposition 4] (see also [7] for a direct definition of the Lie bracket on $C^*(C_X, C_X)$). Alternatively one may use the bijection

$$\operatorname{Hom}_{TV}(TV \otimes (Q \oplus sV), \Lambda(V)) \cong s\Lambda(V) \oplus \operatorname{Der} \Lambda(V)$$

to transfer a Lie algebra structure on $\operatorname{Hom}_{TV}(TV \otimes (Q \oplus sV), \Lambda(V))$ from $s\Lambda(V) \oplus \operatorname{Der} \Lambda(V)$.

Definition 3. Let $L$ be a Lie algebra. An element $x \in L$ is called locally nilpotent if for every $y \in L$, there is a positive integer $k$ such that $(ad x)^k(y) = 0$. A subset $K \subset L$ is called locally nilpotent if each element of $K$ is locally nilpotent.

We deduce from Equation (2) the following

Proposition 4. Let $X$ be a coformal space of homotopy Lie algebra denoted $C_X$. If $X$ has a Quillen model $(\Lambda(V), \delta)$, of length $n$, one has:

1. For $k \neq 1$, $\operatorname{Ext}_A^k(Q, C_X)$ is locally nilpotent,
2. $\operatorname{Ext}_A^1(Q, C_X)$ is a subalgebra of $\operatorname{Ext}_A(Q, C_X)$,
3. If $\operatorname{Ext}_A^1(Q, C_X) = 0$, then $\oplus_{i \geq 1} \operatorname{Ext}_A^i(Q, C_X)$ is an ideal of $\operatorname{Ext}_A(Q, C_X)$, for $i_0 \geq 1$.

We will now assume that $X$ is a coformal 2-cone. Recall that $X$ has a Quillen minimal model of the form $(\Lambda(V_1 \oplus V_2), \delta)$, with $\delta V_1 = 0$ and $\delta V_2 \subset \Lambda(V_1)$. Moreover $\pi_*(\Omega X) \otimes Q = H_*(\Lambda(V_1 \oplus V_2), \delta) = \Lambda(V_1)/I$, where $I$ is the ideal of $\Lambda(V_1)$ generated by $\delta V_2$.

Definition 5. Let $\Lambda(V)$ be a free Lie algebra where $\{a, b, c, \ldots\}$ is a basis of $V$. Denote $\Lambda^n(V)$ the subspace of $\Lambda(V)$ consisting of Lie brackets of length $n$. Consider a basis $\{u_1, u_2, \ldots\}$ of $\Lambda^n(V)$ where each $u_i$ is a Lie monomial. If $x \in \{a, b, c, \ldots\}$, we define the length of $u_i$ in the variable $x$, $l_x(u_i)$, as the number of occurrences of the letter $x$ in $u_i$. If $u = \sum r_i u_i \in \Lambda^n(V)$, define $l_x(u) = \min\{l_x(u_i)\}$ and if $v = \sum v_i$, where $u_i \in \Lambda^1(V)$, $l_x(v) = \min\{l_x(v_i)\}$.

It is straightforward that the above definition extends to the enveloping algebra $T(V)$.

Theorem 6. Let $X$ be a coformal 2-cone and $(\Lambda(V_1 \oplus V_2), \delta)$ be its Quillen minimal model. Choose a basis $\{x_1, x_2, \ldots\}$ for $V_1$ and a basis $\{y_1, y_2, \ldots\}$ for $V_2$. If for some $x_0 \in \{x_1, x_2, \ldots\}$, $l_{x_0}(\delta y_j) \geq 2$ for all $y_j \in \{y_1, y_2, \ldots\}$, then $\operatorname{Ext}_A^*(Q, C_X)$ is infinite dimensional.
Proof. Note that for $i \neq k$ the element $(ad x_i)^n(x_k)$ is a nonzero homology class in $H_4(L(V_1 \oplus V_2), \delta)$ as it contains only one occurrence of $x_k$. Take $y_t \in \{y_1, y_2, \ldots\}$ and $x_m \in \{x_1, x_2, \ldots\}$ with $m \neq k$. For each $n \geq 1$, define $f_n \in \text{Hom}_A(A \otimes sV_2, L_X)$ by $f_n(sy_t) = (ad x_m)^n(x_k)$ and $f_n(sy_j) = 0$ for $j \neq t$. Obviously $f_n \in \text{Hom}_A(A \otimes sV_2, L_X)$ is a cocycle. Suppose that $f_n$ is a coboundary. There exists $g_n \in \text{Hom}_A(A \otimes sV_1, L_X)$ such that $f_n(sy_t) = g_n(ds y_t)$. From the definition of the differential $d$, one has $ds y_t = \sum_i p_i s y_i$, where the $p_i$'s are polynomials in the variables $x_1, x_2, \ldots$. From the hypothesis on the differential $d y_t$ one knows that $l_{s_y}(p_i) \geq 2$ for $i \neq k$ and $l_{s_y}(p_k) \geq 1$. By using the number of occurrences of the variable $x_k$, one deduces from the previous equalities that $(ad x_m)^n(x_k)$ equals the component of length 1 in $x_k$ of $p_k g_n(s x_k)$. Therefore, in the monomial decomposition of $g_n(s x_k)$ (resp. $p_k$), there must exist $(ad x_m)^n(x_k)$ (resp. $x_m^*$). We obtain a contradiction with $l_{s_y}(p_k) \geq 1$.

The cocycles $f_n$ create an infinite number of non-zero classes (of distinct degrees) and the space $\text{Ext}^2_A(Q, L_X)$ is infinite dimensional. \(\square\)

Corollary 7. If hypotheses of the above theorem are satisfied, then $\text{cat}(\Omega B aut_1(X)) = \infty$.

Proof. If $sx \in \text{Ext}^{0,*} \subset L(V_1)/I$ and $f \in \text{Ext}^{2,*}$ then $[f, sx] = \pm sf(x)$. As elements of $\text{Ext}^{2,*}$ vanish on $V_1$, we deduce that $[\text{Ext}^{2,*}, \text{Ext}^{0,*}] = 0$. It follows from Theorem 6 that $J = \text{Ext}^2_{\Lambda}(Q, L_X)$ is an infinite dimensional ideal of $\pi_*(\Omega B aut_1(X))$. Moreover it follows from Equation (2) that $J$ is abelian. We deduce that the category of $B aut_1(X)$ is infinite \cite[Theorem 12.2]{Salvatore}.

If $X$ is an elliptic space of Sullivan minimal model $(\wedge Z, d)$ then $\text{Der} \wedge Z$ is a finite dimensional $Q$-vector space. Hence the homotopy Lie algebra of $B aut_1(X)$ is finite dimensional, therefore $\pi_*(\Omega B aut_1(X)) \otimes Q$ is nilpotent. In \cite{Salvatore}, P. Salvatore proved that if $X = S^{2n+1} \vee S^{2n+1}$, then $\pi_*(\Omega B aut_1(X)) \otimes Q$ contains an element $\alpha$ that is not locally nilpotent. The proof consists in the construction of two outer derivations $\alpha$ and $\beta$ of the free Lie algebra $L(a, b)$, where $|a| = |b| = 2n$, such that $(ad \alpha)^i(\beta) \neq 0$, for every integer $i > 0$. The technique can be applied to any free Lie algebra with at least two generators. Therefore $\pi_*(\Omega B aut_1(X)) \otimes Q$ contains an element $\alpha$ that is not locally nilpotent if $X$ is a wedge of two spheres or more.

P. Salvatore asked if $\pi_*(\Omega B aut_1(X)) \otimes Q$ has always such a property for every hyperbolic space $X$. A positive answer to this question would provide another characterization of the elliptic-hyperbolic dichotomy \cite{Salvatore}.

For a product space we have the following

Proposition 8. If $X = Y \times Z$ is a product space such that the Lie algebra $\pi_*(\Omega B aut_1(Y)) \otimes Q$ is not nilpotent, then $\pi_*(\Omega B aut_1(X)) \otimes Q$ is not nilpotent.

Proof. Let $(\wedge V, d)$ and $(\wedge W, d')$ be Sullivan models of $Y$ and $Z$ respectively. Therefore $(\wedge V \otimes \wedge W, d \otimes d')$ is a Sullivan model of $X$. It follows from \cite{Salvatore} that

$$H_*(\text{Der} (\wedge V \otimes \wedge W)) \cong H_*(\text{Der} \wedge V) \otimes H^*(\wedge V) \oplus H^*(\wedge V) \otimes H_*(\text{Der} \wedge W).$$

Therefore $\pi_*(\Omega B aut_1(Y)) \otimes Q$ is a subalgebra of $\pi_*(\Omega B aut_1(X)) \otimes Q$. \(\square\)


This article may be accessed via WWW at http://www.rmi.acnet.ge/hha/
or by anonymous ftp at

J.-B. Gatsinzi  gatsinzj@mopipi.ub.bw

Department of Mathematics,
University of Botswana,
Private Bag 0022, Gaborone,
Botswana