AN ALGORITHM FOR THE DETERMINATION OF FEEDBACK COEFFICIENTS OF A STATE VARIABLE FEEDBACK CONTROL SYSTEM

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This paper describes a new and simple method of determining the feedback coefficients in a state variable feedback system, given the closed loop transfer function, making use of the transfer functions between different and relevant points in the feedback configuration used to realize the desired closed loop transfer function. Formulas that can be repetitively used to determine the feedback coefficients are derived. The usefulness and simplicity of this method are demonstrated by illustrative examples where the plant has complex poles, coincident poles and zeros and even when the plant is unstable.

1 INTRODUCTION

For an nth order system, there will be n feedback coefficients or gains $k_1,\ldots\ldots k_n$ (k's or k_i s) and since there are n roots of the system, there are enough degrees of freedom to select arbitrarily any desired set of root locations by proper choice of the feedback gains k's [1,3,4].

There are basically two approaches [1] to carry out design when the closed loop transfer function is given. In the first approach, a feedback configuration with undetermined coefficients is chosen. These coefficients are assigned nominal values which are then adjusted so that the resulting closed loop system conforms to the given closed loop transfer function.

The second approach or the usual method , as it is normally referred to, consists of matching the coefficients of different powers of s in the denominators of the given closed loop transfer function and the closed loop transfer function obtained in terms of the feedback gains k's by block diagram manipulation [1]. The calculation of gains using this technique becomes rather tedious and cumbersome when the order of the system is larger than 3. This method is not programmable. The zeros of the plant are not altered in the overall closed loop transfer function when all the states are fed back via constant gains. The pole zero excess of the closed loop system also remains the same as that of the plant in this case. So for an implementable closed loop transfer function, the pole zero excess should be the same as that of the plant.

However, when the system is represented in a state variable form, special canonical forms can be used to calculate these feedback coefficients [3]. An alternative method is Ackermann's formula [3]. Generally speaking Ackermann's formula starts to break down for systems of higher order than 6or 7.

A repetitive procedure is described here to compute the values of feedback coefficients k's when an implementable transfer function is given [2]. In control theory, repetitive methods have the greatest appeal and hence are most sought after [5]. Hence the repetitive procedure described here has the same advantages , apart from the most interesting fact that it is in terms of the relevant transfer functions and in addition that it can be programmed. In other words, it is in the frequency domain and in the time domain. As no method exists [3,5] in the frequency domain which is programmable, no attempt has been made in this paper to program it for comparison purposes. Nonetheless, it has been shown that it can be programmed. The repetitive nature and the simplicity of the method are so obvious that the coefficients can easily be calculated by hand.

Examples have been solved to highlight the procedure even with regard to plants which are unstable, have coincident poles or complex poles.

2 THE ALGORITHM

The overall closed loop transfer function should have a real solution, that is, it should be possible to get a set of values for the feedback coefficients k's from the transfer function. The choice of the overall transfer function should be such that:

1) all its poles are in the left half s-plane (for stability of the system)
2) the same pole zero excess as the plant
3) all its zeros are in the plant.

If condition (2) or (3) is not satisfied , it is impossible to design a feedback configuration with k's. Condition (1) is only to ensure that the closed loop system is stable.

A typical feedback configuration with the blocks to be controlled, the outputs of which are the state variables and with the corresponding k's is shown in Fig.1 The block diagram reduction attempted goes by the name H_{eq} method [1].

If there is positive feedback for the whole system through the feedback coefficient k_1, it is equivalent to the feedback path containing k_1 being removed from the system or just $TF_2(s)$ and $G_1(s)$ in series as illustrated in Fig1

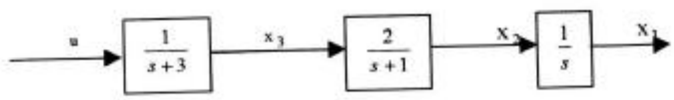

Fig. 1a An example of a plant which is to be controlled

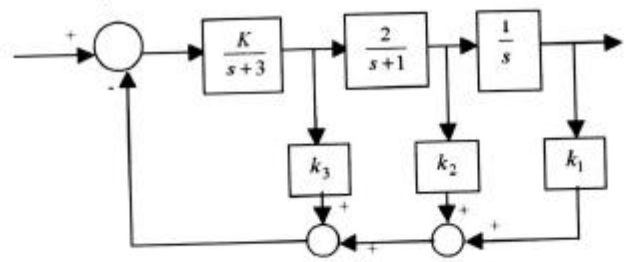

Fig. 1b Feedback configuration used to get the desired closed loop transfer function

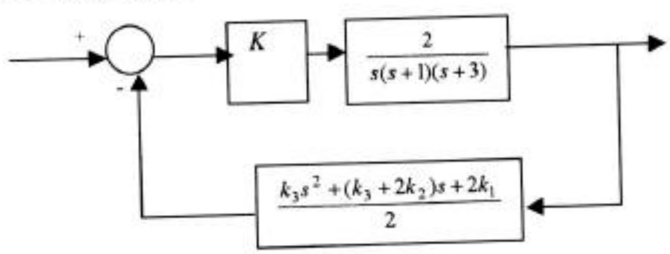

Fig. 1c After block diagram reduction using the most commonly used method

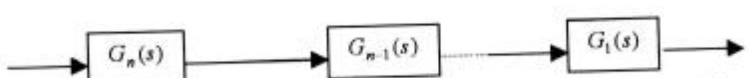

Fig. 2 Arrangement of composite blocks

The composite blocks of the plant to be controlled are denoted by $G_1(s), G_2(s), \ldots, G_n(s)$. These blocks together constitute the plant when cascaded as in Fig.2. Fig. 2 Arrangement of composite blocks which form the plant. The whole system with the feedback configuration is divided into $TF_x(s)$'s (Transfer function) such that the transfer function of the whole configuration is called $TF_1(s)$ and the next with $G_1(s)$ and k_1 removed is called $TF_2(s)$. This is continued till the innermost Loop which is called $TF_n(s)$ and is illustrated in Fig.3 a and Fig.3 b.

So an equation relating to$TF_1(s)$, k_1 and $TF_2(s)$, $G_1(s)$ can be formed.

$$\frac{TF_1(s)}{1 - k_1.TF_1(s)} = TF_2(s).G_1(s) \qquad (1)$$

In this equation only $TF_2(s)$ and k_1 are unknowns.

The pole of $G_1(s)$ is known as it is a composite block of the plant. AT the pole of $G_1(s)$ both sides of the equation (1) equal infinity except when $TF_2(s)$ has a zero at the pole of $G_1(s)$. When this condition is satisfied the denominator of LHS equals zero at the pole of $G_1(s)$ only when no pole of $TF_1(s)$ coincides

with the pole of $G_1(s)$. The equation (2) is derived for the condition when the chosen pole of $G_x(s)$ neither appears as a zero in the preceding blocks nor as a pole in $TF_x(s)$.
Thus

$$[\ 1 - k_1 . TF_1(s)\ ]_{s=\text{pole of } G_1(s)} = 0$$

$$TF_2(s)=\frac{TF_1(s)}{1-k_1.TF_1(s)}\,x\,\frac{1}{G_1(s)} \qquad (3)$$

This can be repeated till all the values for k's are obtained. TF_{n+1} should be equal to the real number, K. (n is the number of blocks in the plant).

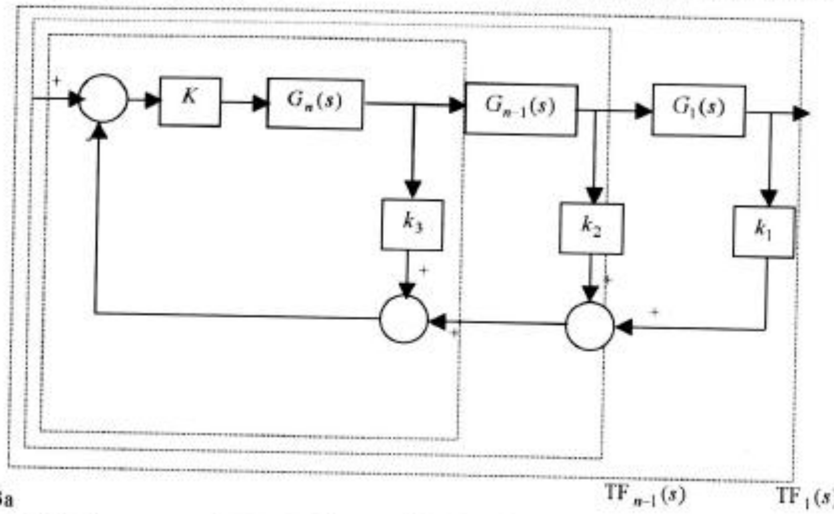

Fig. 3a

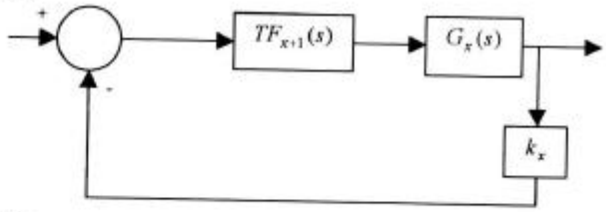

Fig. 3b

Fig. 3 Illustration of the way in which $TF_x(s)$'s are related to $G_x(s)$'s and $k_x(s)$

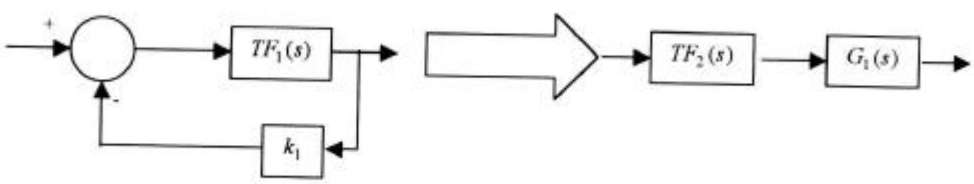

Fig. 4 A positive feedback of k_1 around $TF_1(s)$ is equivalent to $TF_2(s)$ and $G_1(s)$ in series

$$k_1=[1/TF_1(s)]_{s=\text{pole of } G_1(s)} \qquad (2)$$

Rearranging equation(1) we get

$$K=TF_{n+1}(s)=\frac{TF_n(s)}{1-k_n TF_n(s)}\,x\,\frac{1}{G_n(s)} \qquad (4)$$

The formulas to be used repetitively are:

1) $k_x = [1/TF_x(s)]_{s=\text{pole of } G_x(s)}$ (5)

2) $TF_{x+1}(s) = \dfrac{TF_x(s)}{1-k_x TF_x(s)} x \dfrac{1}{G_x(s)}$ (6)

3) $K = TF_{n+1}(s) = \dfrac{TF_n(s)}{1-k_n TF_n(s)} x \dfrac{1}{G_n(s)}$ (7)

The conditions for using equations (5),(6) and (7) are :

1) the pole of $G_x(s)$ that is chosen does not appear as a zero in any of the preceding blocks
2) and also does not appear as a pole of $TF_x(s)$.

This method of solving for k's is easily programmable. A flowchart for the method is given in Fig.5.

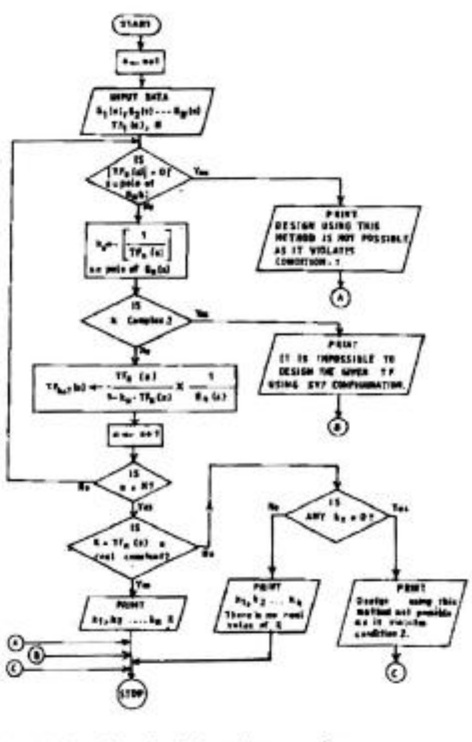

Fig. 5 Flow chart depicting the procedure

3 EXAMPLES

(1) For an unstable plant shown in Fig.6 a which has a

$$TF_1(s) = \frac{80}{[(s+2)^2+4](s+10)}$$

$$k_1 = [1/TF_1(s)]_{s=0} = 1$$

$$TF_2(s) = \frac{TF_1(s)}{1-k_1.TF_1(s)} x \frac{1}{G_1(s)} = \frac{80}{s^2+14s+48}$$

$$k_2 = [1/TF_2(s)]_{s=-2} = 1$$

$$TF_3(s) = \frac{TF_2(s)}{1-k_2.TF_2(s)} x \frac{1}{G_2(s)} = \frac{80}{s+16}$$

$$k_3 = [1/TF_3(s)]_{s=-1} = 3/16$$

$$K = \frac{TF_3(s)}{1-k_3.TF_3(s)} x \frac{1}{G_3(s)} = 80$$

The configuration of the system is shown in Fig.6 b.

(2) When a plant has complex poles as shown in Fig.7 a and a

$$TF_1(s) = \frac{10}{s^4+5s^3+3s^2+10s+10}$$

$$k_1 = [1/TF_1(s)]_{s=0} = 1$$

$TF_2(s)$=

$$\frac{TF_1(s)}{1-k_1.TF_1(s)} x \frac{1}{G_1(s)} = \frac{10}{s^3+5s^2+8s+10}$$

$$k_2 = [1/TF_2(s)]_{s=-1+i} = 0.4$$

$$TF_3(s) = \frac{TF_2(s)}{1-k_2.TF_2(s)} x \frac{1}{G_2(s)} = \frac{10}{s+3}$$

$$k_3 = [1/TF_3(s)]_{s=-1} = 0.2$$

$$K = \frac{TF_3(s)}{[1-k_3.TF_3(s)]} x \frac{1}{G_3(s)} = 10$$

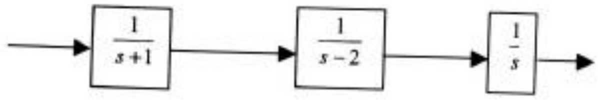

Fig. 6a An unstable plant

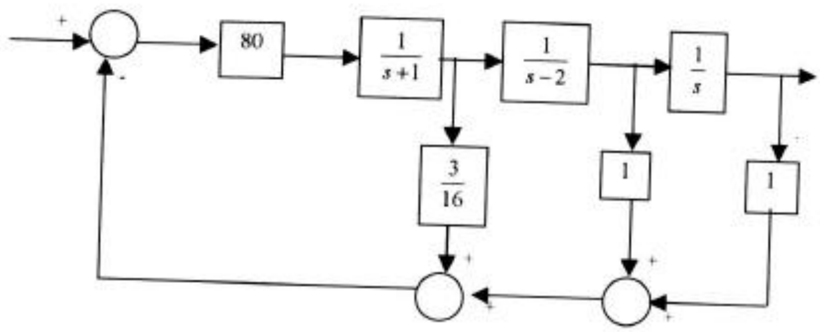

Fig. 6b The feedback configuration

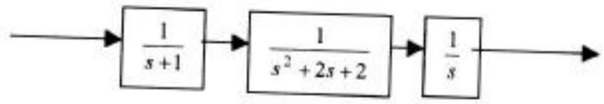

Fig. 7a A plant with complex poles

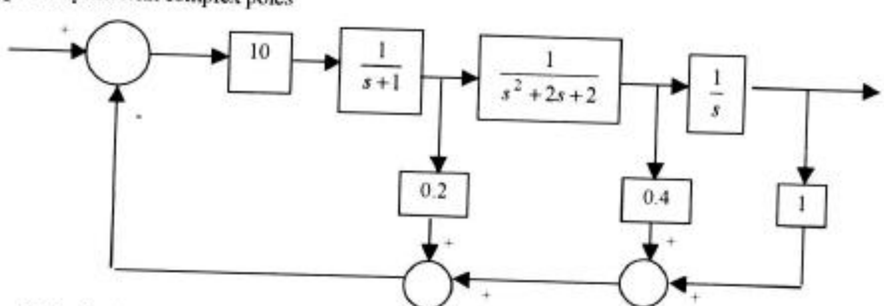

Fig. 7b Feedback configuration

The configuration of the system is shown in Fig.7 b.

(3) For a plant with coincident poles as shown in Fig 8 a and with a closed loop transfer Function

$$TF_1(s) = \frac{10}{s^4 + 12s^3 + 49s^2 + 79s + 51}$$

$$k_1 = [1/TF_1(s)]_{s=-1} = 1$$

$TF_2(s) =$

$$\frac{TF_1(s)}{1 - k_1.TF_1(s)} x \frac{1}{G_1(s)} = \frac{10}{s^3 + 11s^2 + 38s + 41}$$

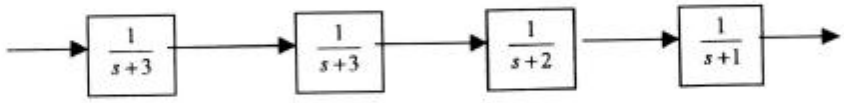

Fig. 8a Plant with coincident poles

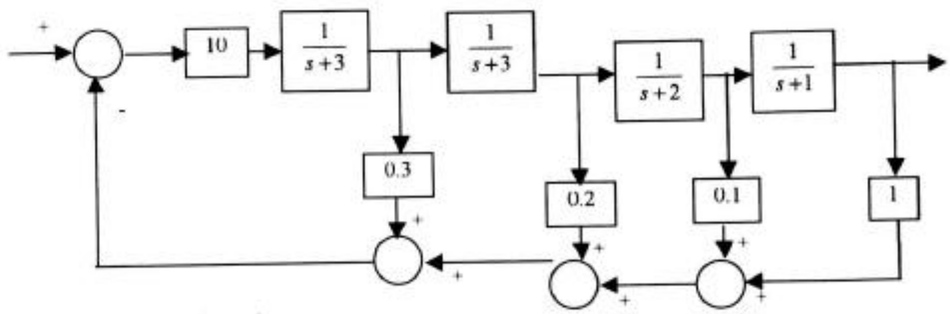

Fig. 8b Feedback configuration

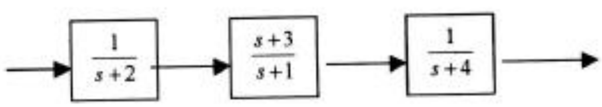

Fig. 9a Plant with a zero

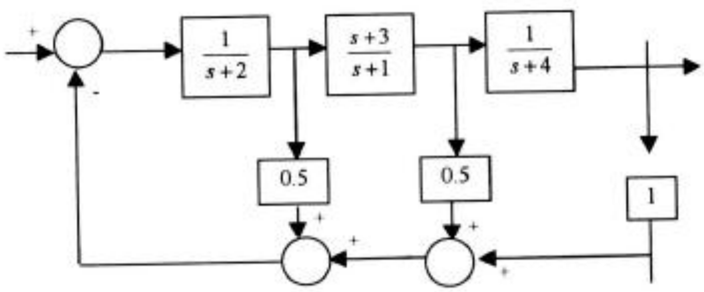

Fig. 9b Feedback configuration

$$k_2 = [1/TF_2(s)]_{s=-2} = 0.1$$

$$TF_3(s) = \frac{TF_2(s)}{1-k_2.TF_2(s)} x \frac{1}{G_2(s)} = \frac{10}{s^2+9s+20}$$

$$k_3 = [1/TF_3(s)]_{s=-3} 0.2$$

$$TF_4(s) = \frac{TF_3(s)}{1-k_3.TF_3(s)} x \frac{1}{G_3(s)} = \frac{10}{s+6}$$

$$k_4 = [1/TF_4(s)]_{s=-3} = 0.3$$

$$K = \frac{TF_4(s)}{1-k_4.TF_4(s)} x \frac{1}{G_4(s)} = 10$$

The closed loop transfer function is shown in Fig. 8 b

(4) For plant which has a zero as shown in Fig. 9 a, the transfer function to be realized is;

$$TF_1(s) = \frac{s+2}{s^3 + 8.1s^2 + 21.7s + 20.2}$$

This transfer function is not feasible because the zero of the transfer function is not contained in the plant. If the transfer function to be realized is

$$TF_1(s) = \frac{s+3}{s^3 + 8.1s^2 + 21.7s + 20}$$

Then

$$k_1 = [1/TF_1(s)]_{s=-4} = 1$$

$$TF_2(s) = \frac{TF_1(s)}{1-k_1.TF_1(s)} x \frac{1}{G_1(s)} = \frac{s+3}{s^2 + 4.1s + 4.3}$$

$$k_2 = [1/TF_2(s)]_{s=-1} = 0.6$$

$$TF_3(s) = \frac{TF_2(s)}{1-k_2.TF_2(s)} x \frac{1}{G_2(s)} = \frac{1}{s+2.5}$$

$$k_3 = [1/TF_3(s)]_{s=-2} = 0.5$$

$$K = \frac{TF_3(s)}{1-k_3.TF_3(s)} x \frac{1}{G_3(s)} = 1$$

The feedback configuration is shown in Fig.9 b.

(5) For a plant shown in Fig.10 a which has a transfer function

$$TF_1(s) = \frac{20}{s^3 + 7s^2 + 14s + 20}$$

The values for k's and K are solved by the usual method of matching the Coefficients and the method described in this paper.

USUAL METHOD

From Fig.10 b and Fig. 10 c, we can get $TF_1(s)$ in terms of k's.

$TF_1(s)$=

$$\frac{2K}{s^3 + (4+Kk_3)s^2 + (3+Kk_3+2Kk_2)s + 2Kk_1}$$

Comparing the coefficients, we get the equations

$2K=20$; $4+Kk_3=7$;

$3+Kk_3+2Kk_2=14$; $2Kk_1=20$

Solving these equations we get the values of k's and K.

$K=10$; $k_1=1$; $k_2=0.4$; $k_3=0.3$

The above steps clearly show that the method is neither easy nor programmable.

Now the method proposed in this paper will be used to calculate the same coefficients.

$$k_1 = [1/TF_1(s)]_{s=0} = 1$$

$$TF_2(s) = \frac{TF_1(s)}{1-k_1.TF_1(s)} x \frac{1}{G_1(s)} = \frac{20}{s^2 + 7s + 14}$$

$$k_2 = [1/TF_2(s)]_{s=-1} = 0.4$$

$$TF_3(s) = \frac{TF_2(s)}{1-k_2.TF_2(s)} x \frac{1}{G_2(s)} = \frac{10}{s+6}$$

$$k_3 = [1/TF_3(s)]_{s=-3} = 0.3$$

$$K = \frac{TF_3(s)}{1-k_3.TF_3(s)} x \frac{1}{G_3(s)} = 10$$

The feedback configuration of the system is as shown in the Fig.10 c with the values of K and k's as determined above.

4 ADVANTAGES OF THE PROPOSED METHOD

A lot of computational effort can be saved if we solve a problem by the method described in this paper rather than by the usual method of matching the coefficients and solving for the required unknowns as clearly demonstrated by the fifth example. The need for drawing block diagrams, block diagram reduction and solving simultaneous equations is eliminated.

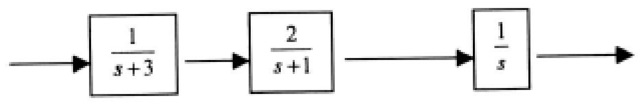

Fig. 10a: Plant with its blocks

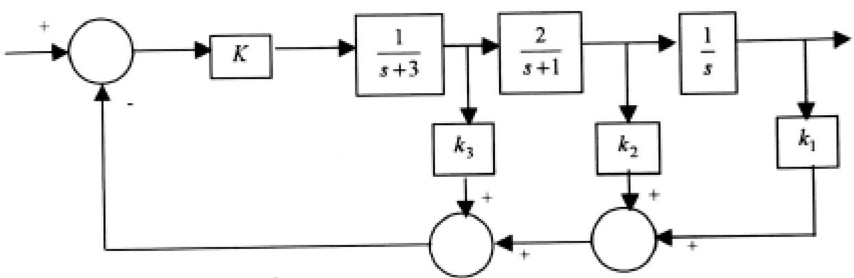

Fig. 10b: Feedback configuration

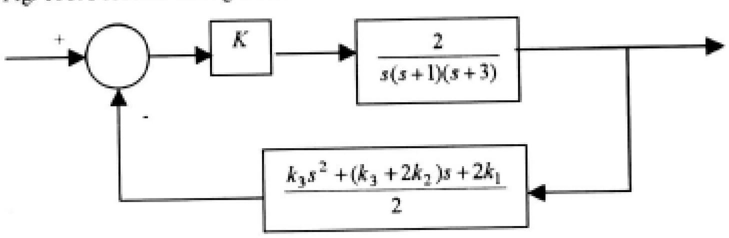

Fig. 10c: After block diagram reduction

The use of these formulas wherein the values for s are substituted to get the values of k's and K is more efficient and systematic and straight forward than the usual method especially when the plant is of higher order.

5 CONCLUSIONS

An algorithm is presented in this paper to calculate the feedback gains in a state variable feedback system , which is simple straight forward and efficient. The simplicity and repetitive use of the formulas developed make the method more appealing from the point of[3] view of determination of the required gains quickly.[4] This is more so with regard to a plant of higher order[5] and thus it is believed that this will be an extremely useful design tool.

6 REFERENCES

1. Melsa, J L and Schultz, D G' "Linear Control Systems", Mcgraw Hill Book Company, New York, 1969.

2 Aurobindo das Gupta, "State Variable feedback and Minor Loop Design" B.Tech. Project Report, Indian Institute of Technology, Madras, India,1988.

3. Franklin, G.; Powell, J D and Naeini, A.E., "Feedback Control of Dynamic Systems" Addison Wesley Publishing Company,1991.

4. Dutton, K.; Thompson, S. and Barraclough, B. "The art of Control Engineering" Addison Wesley Publishing Company,1997.

5. The Theory and Practice gap in Aerospace Control—IEEE Control Systems Special Issue, Vol.19,No.6,Dec 1999.